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ON 2-ADIC ORDERS OF SOME BINOMIAL SUMS

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ABSTRACT. We prove that for any nonnegative integers n and r the binomial sum

$$\sum_{k=-n}^n \binom{2n}{n-k} k^{2r}$$

is divisible by $2^{2n-\min\{\alpha(n), \alpha(r)\}}$, where $\alpha(n)$ denotes the number of 1s in the binary expansion of n . This confirms a recent conjecture of Guo and Zeng [J. Number Theory, **130**(2010), 172–186].

In 1976 Shapiro [3] introduced the Catalan triangle $(\frac{k}{n} \binom{2n}{n-k})_{n \geq k \geq 1}$ and determined the sum of entries in the n th row; namely, he showed that

$$\sum_{k=1}^n k \binom{2n}{n-k} = \frac{n}{2} \binom{2n}{n}.$$

Let $n, r \in \mathbb{N} = \{0, 1, 2, \dots\}$. Recently, Guo and Zeng [1] proved that

$$\frac{2}{n^2 \binom{2n}{n}} \sum_{k=1}^n \binom{2n}{n-k} k^{2r+1}$$

is an odd integer if $n, r \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$. They also conjectured that the binomial sum

$$F(n, r) = \sum_{k=-n}^n \binom{2n}{n-k} k^{2r} \tag{1.1}$$

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is divisible by $2^{2n-\min\{\alpha(n), \alpha(r)\}}$, where $\alpha(n)$ denotes the number of 1s in the binary expansion of n . Note that if $n, r \in \mathbb{Z}^+$ then $F(n, r) = 2 \sum_{k=1}^n \binom{2n}{n-k} k^{2r}$. Actually the conjecture was motivated by Guo and Zeng's following observations:

$$\begin{aligned} \sum_{k=1}^n \binom{2n}{n-k} k^2 &= 2^{2n-2} n, \\ \sum_{k=1}^n \binom{2n}{n-k} k^4 &= 2^{2n-3} n(3n-1), \\ \sum_{k=1}^n \binom{2n}{n-k} k^6 &= 2^{2n-4} n(15n^2 - 15n + 4), \\ \sum_{k=1}^n \binom{2n}{n-k} k^8 &= 2^{2n-5} n(105n^3 - 210n^2 + 147n - 34). \end{aligned}$$

In this paper we shall confirm the sophisticated conjecture of Guo and Zeng. For an integer n and a prime p , the p -adic order of n at p is given by

$$\nu_p(n) = \sup\{v \in \mathbb{N} : p^v \mid n\}.$$

Now we state our main result.

Theorem 1.1. *For any $n, r \in \mathbb{N}$ we have*

$$\nu_2(F(n, r)) \geq 2n - \min\{\alpha(n), \alpha(r)\}, \quad (1.2)$$

where $F(n, r)$ is given by (1.1).

Note that (1.2) can be split into two inequalities:

$$\nu_2(F(n, r)) \geq 2n - \alpha(n) \quad (1.3)$$

and

$$\nu_2(F(n, r)) \geq 2n - \alpha(r). \quad (1.4)$$

In Sections 2 and 3 we will show (1.3) and (1.4) respectively.

2. PROOF OF (1.3)

Let p be any prime. A useful theorem of Legendre (see, e.g., [2, pp. 22–24]) asserts that for any $n \in \mathbb{N}$ we have

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - \alpha_p(n)}{p-1},$$

where $\alpha_p(n)$ is the sum of the digits of n in the expansion of n in base p . In particular, $\nu_2(n!) = n - \alpha(n)$ for all $n = 0, 1, 2, \dots$

Lemma 2.1. (i) For any $n \in \mathbb{Z}^+$ we have

$$\nu_2(n) - 1 = \alpha(n-1) - \alpha(n). \quad (2.1)$$

(ii) Let $s > t \geq 0$ be integers. Then

$$\nu_2\left(\binom{s}{t}\right) \geq \alpha(t) - \alpha(s) + 1. \quad (2.2)$$

Proof. (i) In view of Legendre's theorem, for any positive integer n we have

$$\nu_2(n) = \nu_2(n!) - \nu_2((n-1)!) = n - \alpha(n) - (n-1 - \alpha(n-1)) = \alpha(n-1) - \alpha(n) + 1.$$

This proves (2.1).

(ii) With the help of Legendre's theorem,

$$\begin{aligned} \nu_2\left(\binom{s}{t}\right) &= \nu_2(s!) - \nu_2(t!) - \nu_2((s-t)!) \\ &= s - \alpha(s) - (t - \alpha(t)) - (s-t - \alpha(s-t)) \\ &= \alpha(t) - \alpha(s) + \alpha(s-t) \\ &\geq \alpha(t) - \alpha(s) + 1 \quad (\text{since } s-t \geq 1). \end{aligned}$$

So (2.2) holds. \square

Lemma 2.2. For $n, r \in \mathbb{Z}^+$ we have

$$F(n, r) = n^2 F(n, r-1) - 2n(2n-1)F(n-1, r-1). \quad (2.3)$$

Proof. Since

$$(n^2 - k^2) \binom{2n}{n-k} = 2n(2n-1) \binom{2n-2}{n-1-k},$$

we have

$$\sum_{k=-n}^n \binom{2n}{n-k} k^{2r} = n^2 \sum_{k=-n}^n \binom{2n}{n-k} k^{2r-2} - 2n(2n-1) \sum_{k=-n+1}^{n-1} \binom{2n-2}{n-1-k} k^{2r-2},$$

which gives (2.3). \square

Proof of (1.3). We use induction on $n+r$. Clearly (1.3) holds trivially when $n=0$ or $r=0$.

Now let $n, r \in \mathbb{Z}^+$ and assume (1.3) for any smaller value of $n+r$. By (2.1), (2.3) and the induction hypothesis, we have

$$\begin{aligned} \nu_2(F(n, r)) &\geq \min\{\nu_2(n^2 F(n, r-1)), \nu_2(2n(2n-1)F(n-1, r-1))\} \\ &= \min\{2\nu_2(n) + \nu_2(F(n, r-1)), 1 + \nu_2(n) + \nu_2(F(n-1, r-1))\} \\ &\geq \min\{2\nu_2(n) + 2n - \alpha(n), 1 + \nu_2(n) + 2(n-1) - \alpha(n-1)\} \\ &= 2n - \alpha(n). \end{aligned}$$

This concludes the induction step. \square

3. PROOF OF (1.4)

Lemma 3.1. *For $n, r \in \mathbb{Z}^+$ we have*

$$\begin{aligned} F(n, r) = & 4F(n-1, r) - \sum_{i=0}^{r-1} \binom{2r}{2i} F(n, i) - 2(2n-1) \sum_{i=0}^{r-1} \binom{2r}{2i+1} F(n-1, i) \\ & + n \sum_{i=0}^{r-1} \binom{2r}{2i+1} F(n, i) + 2 \sum_{i=0}^{r-1} \binom{2r}{2i} F(n-1, i). \end{aligned} \quad (3.1)$$

Proof. Let $n \in \mathbb{N}$ and $r \in \mathbb{Z}^+$. We want to prove (3.1) with n in it replaced by $n+1$.

Clearly

$$\begin{aligned} F(n, r) &= \sum_{k=-n-1}^{n-1} \binom{2n}{n-1-k} (k+1)^{2r} \\ &= \sum_{k=-n-1}^n \left(\binom{2n+1}{n-k} - \binom{2n}{n-k} \right) \sum_{j=0}^{2r} \binom{2r}{j} k^j \\ &= \sum_{j=0}^{2r} \binom{2r}{j} \sum_{k=-n-1}^n \binom{2n+1}{n-k} k^j - \sum_{i=0}^r \binom{2r}{2i} \sum_{k=-n}^n \binom{2n}{n-k} k^{2i}, \end{aligned} \quad (3.2)$$

in the last step we use the fact that if j is odd then

$$\begin{aligned} \sum_{k=-n}^n \binom{2n}{n-k} k^j &= \frac{1}{2} \left(\sum_{k=-n}^n \binom{2n}{n-k} k^j + \sum_{k=-n}^n \binom{2n}{n+k} (-k)^j \right) \\ &= \sum_{k=-n}^n \binom{2n}{n-k} (k^j + (-k)^j) = 0. \end{aligned}$$

When j is even, we have

$$\begin{aligned} 2 \sum_{k=-n-1}^n \binom{2n+1}{n-k} k^j &= \sum_{k=-n-1}^{n+1} \left(\binom{2n+1}{n-k} + \binom{2n+1}{n+k} \right) k^j \\ &= \sum_{k=-n-1}^{n+1} \binom{2n+2}{n+1-k} k^j = F\left(n+1, \frac{j}{2}\right). \end{aligned} \quad (3.3)$$

If j is odd, then

$$\begin{aligned} & (n+1) \sum_{k=-n-1}^n \binom{2n+1}{n-k} k^{j-1} + \sum_{k=-n-1}^n \binom{2n+1}{n-k} k^j \\ &= \sum_{k=-n-1}^n (n+1+k) \binom{2n+1}{n-k} k^{j-1} = (2n+1) \sum_{k=-n}^n \binom{2n}{n-k} k^{j-1}, \end{aligned}$$

i.e.,

$$\begin{aligned} \sum_{k=-n-1}^n \binom{2n+1}{n-k} k^j &= (2n+1) \sum_{k=-n}^n \binom{2n}{n-k} k^{j-1} - (n+1) \sum_{k=-n-1}^n \binom{2n+1}{n-k} k^{j-1} \\ &= (2n+1) F\left(n, \frac{j-1}{2}\right) - \frac{n+1}{2} F\left(n+1, \frac{j-1}{2}\right), \end{aligned} \quad (3.4)$$

where we use (3.3) in the last step. Combining (3.2)-(3.4), we get

$$\begin{aligned} F(n, r) &= \frac{1}{2} \sum_{i=0}^r \binom{2r}{2i} F(n+1, i) + (2n+1) \sum_{i=0}^{r-1} \binom{2r}{2i+1} F(n, i) \\ &\quad - \frac{n+1}{2} \sum_{i=0}^{r-1} \binom{2r}{2i+1} F(n+1, i) - \sum_{i=0}^r \binom{2r}{2i} F(n, i), \end{aligned}$$

which yields the desired result. \square

Proof of (1.4). We still use induction on $n+r$. There is nothing to do if $n=0$ or $r=0$. Assume that $n, r \geq 1$ and (1.4) holds for any smaller value of $n+r$. In view of Lemma 3.1, $\nu_2(F(n, r))$ is not smaller than the minimum of the following numbers:

$$\begin{aligned} &2 + \nu_2(F(n-1, r)), \min_{0 \leq i < r} \nu_2\left(\binom{2r}{2i} F(n, i)\right), \min_{0 \leq i < r} \nu_2\left(n \binom{2r}{2i+1} F(n, i)\right) \\ &1 + \min_{0 \leq i < r} \nu_2\left(\binom{2r}{2i+1} F(n-1, i)\right), 1 + \min_{0 \leq i < r} \nu_2\left(\binom{2r}{2i} F(n-1, i)\right). \end{aligned}$$

By the induction hypothesis and Lemma 2.1(ii), we have $\nu_2(F(n-1, r)) \geq 2n-2-\alpha(r)$, and also

$$\begin{aligned} \nu_2\left(\binom{2r}{2i} F(n, i)\right) &\geq 2n - \alpha(i) + \alpha(2i) - \alpha(2r) + 1 = 2n - \alpha(r) + 1, \\ \nu_2\left(\binom{2r}{2i+1} F(n-1, i)\right) &\geq 2n - 2 - \alpha(i) + \alpha(2i+1) - \alpha(2r) + 1 = 2n - \alpha(r), \\ \nu_2\left(n \binom{2r}{2i+1} F(n, i)\right) &\geq 2n - \alpha(i) + \alpha(2i+1) - \alpha(2r) + 1 = 2n - \alpha(r) + 2, \end{aligned}$$

and

$$\nu_2\left(\binom{2r}{2i} F(n-1, i)\right) \geq 2n - 2 - \alpha(i) + \alpha(2i) - \alpha(2r) + 1 = 2n - \alpha(r) - 1.$$

Thus (1.4) follows. \square

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